Hierarchical Mean-Field Theories in Quantum Statistical Mechanics

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(Dated: Received February 6, 2008)

We present a theoretical framework and a calculational scheme to study the coexistence and competition of thermodynamic phases in quantum statistical mechanics. The crux of the method is the realization that the microscopic Hamiltonian, modeling the system, can always be written in a hierarchical operator language that unveils all symmetry generators of the problem and, thus, possible thermodynamic phases. In general one cannot compute the thermodynamic or zero-temperature properties exactly and an approximate scheme named "hierarchical mean-field approach" is introduced. This approach treats all possible competing orders on an equal footing. We illustrate the methodology by determining the phase diagram and quantum critical point of a bosonic lattice model which displays coexistence and competition between antiferromagnetism and superfluidity.

PACS numbers: 05.30.-d, 05.70.Fh, 05.30.Jp, 75.10.Jm

During last decade we have witnessed great advances in materials synthesis and fabrication. The rate at which new compounds with multiplicity of distinct phases and characteristic functionalities are generated has outpaced our complete physical understanding of the fundamental principles behind such complex phenomena. For example, whether the mechanism controlling the coexistence and/or competition between magnetism and superconductivity or Bose-Einstein condensation has the same physical origin in different classes of materials is still an open question [1]. On the other hand, the possibility of control and tunability of the interactions of the elementary constituents (i.e, quantum control) offers the potential to design new states of matter with unforeseen applications [2]. Despite great theoretical advances there is a lack of a systematic and reliable methodology to study and predict the behavior of these complex systems. It is the main purpose of this paper to present a promising step in that direction.

The modern theory of phase transitions starts with Landau's pioneering work in 1937 [3]. One of his achievements was the realization of the fundamental relation between spontaneous symmetry breaking and the order parameter (OP) that measures this violation, thus giving simple prescriptions to describe order in terms of irreducible representations of the symmetry group involved. Another was the development of a phenomenological calculational scheme to study the behavior of systems near a phase transition. Landau's theory has been successfully applied to study phase transitions where thermal fluctuations are most relevant. Certainly, the theory was not designed to study zero-temperature (quantum) phase transitions, i.e., the qualitative changes of the macroscopic state of the system induced by tuning parameters of its Hamiltonian.

In the quantum description of matter, a physical system is naturally associated with a *language* of operators [4]. In previous work [5, 6] we outlined a framework to identify OPs based upon isomorphic mappings to a *hierarchical language* (HL) defined by the set of operators which in the fundamental representation (of dimension D) has the largest number of symmetry generators of the group. *Any* local operator can be expressed as a *linear* combination of the generators of the HL. The building of the HL depends upon the dimension D of the local Hilbert space modeling the physical phenomena one is investigating. For instance, if one is modeling a doped antiferromagnetic (AF) insulator with a t-J Hamiltonian [7], then D=3 (i.e., there are three possible states per site) and a HL is generated by a basis of $su(3)$ in the fundamental representation [5, 6]. As explained and proved in Refs. [6, 8], there is always a HL associated to each physical problem. These ideas complement Landau's concept of an OP providing a mechanism to reveal them, something that is outside the groundwork of Landau's theory. Indeed, this theory does not say what the OPs should be in a general situation.

It turns out that these isomorphic mappings not only unveil hidden symmetries of the original physical system but also manifestly establish equivalences between seemingly unrelated physical phenomena. Nonetheless, this is not sufficient to determine the *exact* phase diagram of the problem: One has to resort to either numerical simulations with their well-known limitations or, as will be shown in the present paper, to a *guided* approximation which at least preserves the qualitative nature of the possible thermodynamic states. A key observation in this regard is the fact that typical model Hamiltonian operators written in the HL become quadratic in the symmetry generators of the hierarchical group, and this result is independent of the group of symmetries of the Hamiltonian.

This latter result suggests a simple approximation, based upon group theoretical grounds, which deals with competing orders on an equal footing and will be termed *hierarchical mean-field theory* (HMFT). In a sense, that will become clear below, HMFT constitutes the *optimum* mean-field (MF) or saddle-point solution that approximates the energy and correlation functions of the original problem. The HMFT is distinctly suitable when the various phases displayed by a system are the result of competing interactions and non-linear couplings of their constituents matter fields. From the theoretical standpoint these systems are strongly correlated since no obvious small coupling constant exists, as a consequence they exhibit high sensitivity to small parameter changes. It is then clear the importance of developing a methodology that treats all possible competing orders on an equal footing.

We will now illustrate the methodology by example and determine the zero temperature phase diagram of a simple model which displays coexistence and competition between antiferromagnetism and Bose-Einstein condensation (superfluidity). The model represents a gas of interacting two-flavor ($\sigma =\uparrow,\downarrow$)

hard-core bosons with Hamiltonian ($t > 0$)

$$H = t \sum_{\langle \mathbf{i},\mathbf{j}\rangle,\sigma} \left(\bar{b}^\dagger_{\mathbf{i}\sigma} \bar{b}_{\mathbf{j}\sigma} + \text{H.c.} \right) + J \sum_{\langle \mathbf{i},\mathbf{j}\rangle} (\mathbf{s_i} \cdot \mathbf{s_j} - \frac{\bar{n}_\mathbf{i} \bar{n}_\mathbf{j}}{4}) + V \sum_{\langle \mathbf{i},\mathbf{j}\rangle} \bar{n}_\mathbf{i} \bar{n}_\mathbf{j} - \bar{\mu} \sum_\mathbf{j} \bar{n}_\mathbf{j} \, , \quad (1)$$

where $\langle \mathbf{i},\mathbf{j}\rangle$ stands for nearest-neighbor sites (bond) in an otherwise regular N_s-sites lattice of coordination z and $D = 3$. The number operator $\bar{n}_\mathbf{j} = \bar{n}_{\mathbf{j}\uparrow} + \bar{n}_{\mathbf{j}\downarrow}$ ($\bar{n}_{\mathbf{j}\sigma} = \bar{b}^\dagger_{\mathbf{j}\sigma} \bar{b}_{\mathbf{j}\sigma}$), and $\mathbf{s_j} = \frac{1}{2} \bar{b}^\dagger_{\mathbf{j}\mu} \boldsymbol{\sigma}_{\mu\nu} \bar{b}_{\mathbf{j}\nu}$ is a $s = \frac{1}{2}$ operator ($\boldsymbol{\sigma}$ denoting Pauli matrices). The algebra satisfied by the hard-core bosons is [5]: $[\bar{b}_{\mathbf{i}\sigma}, \bar{b}_{\mathbf{j}\sigma'}] = 0$, $[\bar{b}_{\mathbf{i}\sigma}, \bar{b}^\dagger_{\mathbf{j}\sigma'}] = \delta_{\mathbf{ij}}(1 - 2\bar{n}_{\mathbf{i}\sigma} - \bar{n}_{\mathbf{i}\bar{\sigma}})$ (if $\sigma = \sigma'$), or $-\delta_{\mathbf{ij}} \bar{b}^\dagger_{\mathbf{i}\sigma'} \bar{b}_{\mathbf{i}\sigma}$ (if $\sigma \neq \sigma'$). Notice that H is an extended t-J-like model of hard-core bosons instead of constrained fermions [10]. These hard-core bosons could represent three-state atoms, like the ones used in trapped Bose-Einstein condensates, moving in an optical lattice potential. For the sake of clarity we will only consider the AF case $J > 0$.

As explained in the introduction the first step in determining its phase diagram consists of re-writing H in a HL. The latter is realized by $SU(3)$-spin generators in the fundamental representation, and its mapping to the hard-core boson language can be compactly written as [5]

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_\mathbf{j} & \bar{b}_{\mathbf{j}\uparrow} & \bar{b}_{\mathbf{j}\downarrow} \\ \bar{b}^\dagger_{\mathbf{j}\uparrow} & \bar{n}_{\mathbf{j}\uparrow} - \frac{1}{3} & \bar{b}^\dagger_{\mathbf{j}\uparrow} \bar{b}_{\mathbf{j}\downarrow} \\ \bar{b}^\dagger_{\mathbf{j}\downarrow} & \bar{b}^\dagger_{\mathbf{j}\downarrow} \bar{b}_{\mathbf{j}\uparrow} & \bar{n}_{\mathbf{j}\downarrow} - \frac{1}{3} \end{pmatrix} . \quad (2)$$

The three components $s^z_\mathbf{j} = (\bar{n}_{\mathbf{j}\uparrow} - \bar{n}_{\mathbf{j}\downarrow})/2$, $s^+_\mathbf{j} = \bar{b}^\dagger_{\mathbf{j}\uparrow} \bar{b}_{\mathbf{j}\downarrow}$ and $s^-_\mathbf{j} = \bar{b}^\dagger_{\mathbf{j}\downarrow} \bar{b}_{\mathbf{j}\uparrow}$ generate the spin $su(2)$ subalgebra, i.e., they are the components of the local magnetization. The five additional components correspond to the Bose-Einstein condensate and the charge density wave local OPs. In the HL, H represents a Heisenberg-like Hamiltonian [9] in the presence of an external magnetic field μ' ($J_{\mu\nu} = J_{\nu\mu}$)

$$H = \sum_{\langle \mathbf{i},\mathbf{j}\rangle} J_{\mu\nu} \mathcal{S}^{\mu\nu}(\mathbf{i}) \mathcal{S}^{\nu\mu}(\mathbf{j}) - \mu' \sum_\mathbf{j} \mathcal{S}^{00}(\mathbf{j}) \, , \quad (3)$$

with $J_{00} = V - J/2$, $J_{01} = J_{02} = t$, $J_{11} = J_{12} = J_{22} = J/2$, and $\mu' = \frac{\mathsf{z}}{3}(2V - J/2) - \bar{\mu}$. Note that through the mapping we transformed an interacting problem into another problem that is quadratic in the basis of the algebra $su(3)$ in the fundamental representation, but which is not necessarily $SU(3)$ symmetric. This HL furnishes the natural framework to analyze the symmetries of the Hamiltonian H. There is always an $SU(2)$ spin symmetry generated by $\mathcal{S}^{11} - \mathcal{S}^{22}$, $\mathcal{S}^{12}$, and $\mathcal{S}^{21}$. When $\mu' = 0$ and $V = 2t$, there are five additional generators of symmetries related to the charge degrees of freedom. Moreover, if $J = V = 2t$ there is full $SU(3)$ symmetry. For $\mu' \neq 0$, the only charge symmetry that remains is a $U(1)$ symmetry generated by $\mathcal{S}^{00}$ (conservation of the total charge). In this way the HL, leading to a unique OP from which all possible embedded orderings are derived, provides a unified description of the possible thermodynamic states of the system. Yet, it remains to establish the orderings that survive as a result of tuning the parameters of the Hamiltonian or external variables such as temperature and filling.

For arbitrary values of the parameters $J/t, V/t$, we do not know a priori how to determine exactly the phase diagram of H [11]. The idea behind the HMFT is to perform an approximation which deals with all possible local OPs on an equal footing with no privileged *symmetry axes* and, hopefully, retains the qualitative topology of the phase diagram. With H written in the HL one immediately realizes that the simplest HMFT can be achieved if we re-write H in terms of $SU(3)$ Schwinger-Wigner (SW) bosons (3 flavors $\alpha = \downarrow, 0, \uparrow$) [7]. The mapping is expressed as

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} n_{\mathbf{j}0} - \frac{1}{3} & b^\dagger_{\mathbf{j}0} b_{\mathbf{j}\uparrow} & b^\dagger_{\mathbf{j}0} b_{\mathbf{j}\downarrow} \\ b^\dagger_{\mathbf{j}\uparrow} b_{\mathbf{j}0} & n_{\mathbf{j}\uparrow} - \frac{1}{3} & b^\dagger_{\mathbf{j}\uparrow} b_{\mathbf{j}\downarrow} \\ b^\dagger_{\mathbf{j}\downarrow} b_{\mathbf{j}0} & b^\dagger_{\mathbf{j}\downarrow} b_{\mathbf{j}\uparrow} & n_{\mathbf{j}\downarrow} - \frac{1}{3} \end{pmatrix} , \quad (4)$$

with the SW bosons $b^\dagger_{\mathbf{j}\alpha}$ satisfying the constraint $n_{\mathbf{j}\downarrow} + n_{\mathbf{j}0} + n_{\mathbf{j}\uparrow} = 1$. The resulting Hamiltonian ($V = 2t$ with no loss of generality) is [9]

$$H = -\sum_{\langle \mathbf{i},\mathbf{j}\rangle} (\frac{J}{2} A^\dagger_{\mathbf{ij}} A_{\mathbf{ij}} + t \sum_{\sigma=\uparrow,\downarrow} B^\dagger_{\sigma\mathbf{ij}} B_{\sigma\mathbf{ij}}) - \mu \sum_\mathbf{j} n_{\mathbf{j}0} \, , \quad (5)$$

where $\mu = \mathsf{z}t - \bar{\mu}$ and the ordering operators

$$\begin{cases} A^\dagger_{\mathbf{ij}} = b^\dagger_{\mathbf{i}\uparrow} b^\dagger_{\mathbf{j}\downarrow} - b^\dagger_{\mathbf{i}\downarrow} b^\dagger_{\mathbf{j}\uparrow} \\ B^\dagger_{\sigma\mathbf{ij}} = b^\dagger_{\mathbf{i}\sigma} b^\dagger_{\mathbf{j}0} - b^\dagger_{\mathbf{i}0} b^\dagger_{\mathbf{j}\sigma} \end{cases} ,$$

which transform as singlets with respect to the generators of $SU(2)$ spin and charge symmetries, respectively. In other words, $[A^\dagger_{\mathbf{ij}}, \mathcal{S}^{12(21)}(\mathbf{i}) + \mathcal{S}^{12(21)}(\mathbf{j})] = 0 = [B^\dagger_{\uparrow(\downarrow)\mathbf{ij}}, \mathcal{S}^{10(20)}(\mathbf{i}) + \mathcal{S}^{10(20)}(\mathbf{j})]$.

Since the $su(N)$ languages provide a complete set of HLs, any model Hamiltonian can be written in a similar fashion once we identify the appropriate HL and apply the corresponding SW mapping in the *fundamental representation* (the ordering operators will, of course, have a different meaning and algebraic expressions). The key point is that the Hamiltonian operator in the HL becomes quadratic in the symmetry generators of the hierarchical group.

The idea behind any MF approximation is to disentangle interaction terms into quadratic ones replacing some of the elementary mode operators by their mean value. The crux of our HMFT is that the approximation is done in the HL where all possible local OPs are treated on an equal footing and the number of operators replaced by their mean value is minimized since the Hamiltonian is quadratic in the symmetry generators. In this way, the information required is minimal. In mathematical terms, given $\mathcal{O}^\dagger_{\mathbf{ij}} \mathcal{O}_{\mathbf{ij}} = \langle \mathcal{O}^\dagger_{\mathbf{ij}} \rangle \mathcal{O}_{\mathbf{ij}} + \mathcal{O}^\dagger_{\mathbf{ij}} \langle \mathcal{O}_{\mathbf{ij}} \rangle - \langle \mathcal{O}^\dagger_{\mathbf{ij}} \rangle \langle \mathcal{O}_{\mathbf{ij}} \rangle + (\mathcal{O}^\dagger_{\mathbf{ij}} - \langle \mathcal{O}^\dagger_{\mathbf{ij}} \rangle)(\mathcal{O}_{\mathbf{ij}} - \langle \mathcal{O}_{\mathbf{ij}} \rangle)$, for an arbitrary bond-operator $\mathcal{O}_{\mathbf{ij}}$, the approximation amounts to neglect the latter fluctuations, i.e., $\mathcal{O}^\dagger_{\mathbf{ij}} \mathcal{O}_{\mathbf{ij}} \approx \langle \mathcal{O}^\dagger_{\mathbf{ij}} \rangle \mathcal{O}_{\mathbf{ij}} + \mathcal{O}^\dagger_{\mathbf{ij}} \langle \mathcal{O}_{\mathbf{ij}} \rangle - \langle \mathcal{O}^\dagger_{\mathbf{ij}} \rangle \langle \mathcal{O}_{\mathbf{ij}} \rangle$ [12]. An important result is that all local OPs are equally treated and, moreover, symmetries of the original Hamiltonian related to the OPs are not broken explicitly in certain limits.

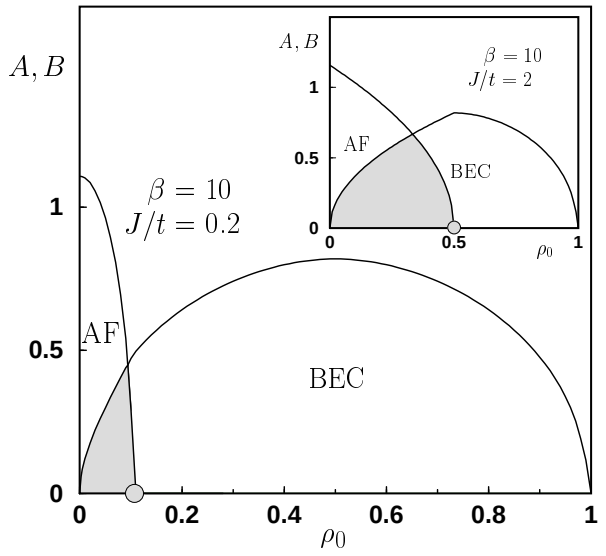

FIG. 1: Order parameters as a function of the density ρ_0 for different values of J/t and inverse temperature $\beta = 10$ (in units of t^{-1}). The filled circle on the density axis indicates a quantum critical point.

The resulting MF Hamiltonian together with the SW-boson constraint (with Lagrange multiplier λ) $\tilde{H} = H_{MF} + \lambda \sum_{\mathbf{j}} (n_{\mathbf{j}\downarrow} + n_{\mathbf{j}0} + n_{\mathbf{j}\uparrow})$ reads [9]

$$\begin{aligned}\tilde{H} &= -\sum_{\langle \mathbf{i},\mathbf{j}\rangle} [\frac{JA}{2}(A^{\dagger}_{\mathbf{ij}} + A_{\mathbf{ij}}) + tB \sum_{\sigma=\uparrow,\downarrow} (B^{\dagger}_{\sigma\mathbf{ij}} + B_{\sigma\mathbf{ij}})] \\ &\quad -\mu \sum_{\mathbf{j}} n_{\mathbf{j}0} + \lambda \sum_{\mathbf{j}} (n_{\mathbf{j}\downarrow} + n_{\mathbf{j}0} + n_{\mathbf{j}\uparrow}) \\ &= \sum_{\mathbf{k}\in \mathrm{RBZ}} [\Lambda_A \, b^{\dagger}_{\mathbf{k}\uparrow} b^{\dagger}_{-\mathbf{k}+\mathbf{Q}\downarrow} + \Lambda_B \, (b^{\dagger}_{\mathbf{k}\uparrow} b^{\dagger}_{-\mathbf{k}+\mathbf{Q}0} + \\ &\quad b^{\dagger}_{\mathbf{k}\downarrow} b^{\dagger}_{-\mathbf{k}+\mathbf{Q}0}) + \mathrm{H.c.}] + (\lambda - \mu) n_{\mathbf{k}0} + \lambda \sum_{\sigma=\uparrow,\downarrow} n_{\mathbf{k}\sigma} \qquad (6)\end{aligned}$$

where the sum of momenta $\mathbf{k}$ is performed over the reduced Brillouin zone (RBZ) with AF ordering wave vector $\mathbf{Q}$ and $n_{\mathbf{k}\alpha} = b^{\dagger}_{\mathbf{k}\alpha} b_{\mathbf{k}\alpha}$, with $b^{\dagger}_{\mathbf{k}\alpha}$ representing Fourier transformed modes. $\Lambda_A = -2JA\gamma_{\mathbf{k}}$, $\Lambda_B = -4tB\gamma_{\mathbf{k}}$, with $\gamma_{\mathbf{k}} = \frac{1}{z}\sum_{\boldsymbol{\delta}} e^{i\mathbf{k}\cdot\boldsymbol{\delta}}$ ($\boldsymbol{\delta}$ are nearest-neighbor vectors). Note that when $B = 0$ in H_{MF}, the $SU(2)$ spin and $U(1)$, $\mathcal{S}^{00}$, symmetries are conserved; the opposite case $A = 0$ preserves $\mathcal{S}^{10(01)} + \mathcal{S}^{20(02)}$ and $\mathcal{S}^{11} + \mathcal{S}^{22} - \mathcal{S}^{00}$ symmetries. In Eq. (6) we have only considered homogeneous solutions [14].

The corresponding self-consistent MF equations to solve are

$$\begin{cases} A = \dfrac{8}{zN_s} \displaystyle\sum_{\mathbf{k}\in \mathrm{RBZ}} \gamma_{\mathbf{k}} \langle b^{\dagger}_{\mathbf{k}\uparrow} b^{\dagger}_{-\mathbf{k}+\mathbf{Q}\downarrow} \rangle_{MF} \, , \\ B = \dfrac{8}{zN_s} \displaystyle\sum_{\mathbf{k}\in \mathrm{RBZ}} \gamma_{\mathbf{k}} \langle b^{\dagger}_{\mathbf{k}\sigma} b^{\dagger}_{-\mathbf{k}+\mathbf{Q}0} \rangle_{MF} \, , \\ 1 = \dfrac{1}{N_s} \displaystyle\sum_{\mathbf{k}\in \mathrm{RBZ}} \sum_{\alpha} \langle n_{\mathbf{k}\alpha} \rangle_{MF} \, . \end{cases}$$

We are thus left with a non-interacting system of SW bosons. Now we follow Colpa [15] and diagonalize para-unitarily the Hamiltonian matrix $\tilde{H}$ [16]. The application of a homogeneous linear transformation leads to [9]

$$\tilde{H} = \sum_{\mathbf{k}\in \mathrm{RBZ}} \sum_{i=0}^{5} \omega_{i\mathbf{k}} \, \alpha^{\dagger}_{i\mathbf{k}} \alpha_{i\mathbf{k}} \, , \qquad (7)$$

where the mode energies $\omega_{i\mathbf{k}}$ are, at least, two-fold degenerate [17]. In Fig. 1, we display the orders A and B as a function of $\rho_0 = \frac{1}{N_s}\sum_{\mathbf{j}} \langle n_{\mathbf{j}0} \rangle$ at very low temperature ($\beta = 10$) and different ratios of the competing interactions J/t for a two-dimensional lattice [18]. The relation between the OPs of the original problem, Eq. (1), and A and B is given by $\frac{1}{N_s^2}\sum_{\mathbf{i},\mathbf{j}} e^{i\mathbf{Q}\cdot(\mathbf{r_i}-\mathbf{r_j})} \langle \bar{b}^{\dagger}_{\mathbf{i}\sigma} \bar{b}_{\mathbf{j}\sigma'} \rangle_{MF} \propto B^2$, $\frac{1}{N_s^2}\sum_{\mathbf{i},\mathbf{j}} e^{i\mathbf{Q}\cdot(\mathbf{r_i}-\mathbf{r_j})} \langle s^{+}_{\mathbf{i}} s^{-}_{\mathbf{j}} \rangle_{MF} \propto A^2$, and justifies the labeling of the phases AF (antiferromagnet) and BEC (Bose-Einstein condensate) in Fig. 1. A way to qualitatively understand this quantum phase diagram is to look into the OP space as displayed in Fig. 2. As ρ_0 (or chemical potential) varies from 0 to 1, the OP (depicted as an arrow) moves in OP space. When $\rho_0 = 0$ the order is purely AF and the arrow lies on the $\uparrow\downarrow$ axis. For $\rho_0 \neq 0$, the arrow has projections onto the 3 $su(2)$ axis, i.e., the AF state *coexists* with a BEC state. There is a particular critical value of $\rho_0 = \rho_{0c} < 1$ for which the AF ordering vanishes and the OP is purely BEC with the arrow lying in the $\uparrow 0 - \downarrow 0$ plane. This BEC ordering persists until $\rho_0 = 1$, where it vanishes.

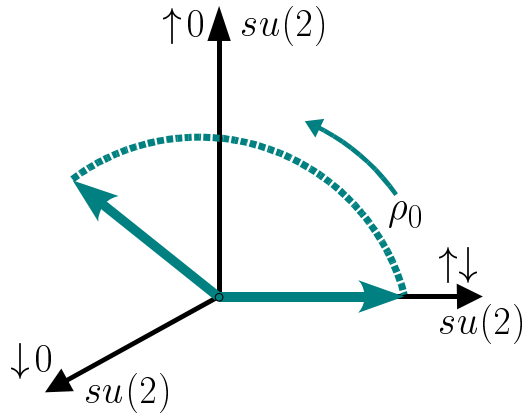

FIG. 2: Schematics of the order parameter change (as a function of ρ_0) with the three $su(2)$ axis describing 3 different $su(2)$ subalgebras of $su(3)$. Note that the plane $\uparrow 0 - \downarrow 0$ represents the 5 charge symmetry generators while the $\uparrow\downarrow$ axis is associated to the remaining three generators of magnetism.

As can be inferred from our presentation, there are two complementary aspects to studying competition and coexistence between phase orderings in strongly coupled quantum systems. One is the direct discovery of the *hidden unity* and subsequent determination of the possible phases and their transitions, given a Hamiltonian operator modeling the complex material of interest. This is the aspect we have described

in the present paper. The second aspect, to be discussed in a separate publication, involves the design or engineering of new states of matter using the inverse path of logic. Essentially, the idea consists of tailoring effective Hamiltonians based upon a general symmetry analysis of the possible orderings one would like to realize at zero temperature. Tuning the parameters of these symmetry-based effective Hamiltonians allows one to move in parameter space along the previously established orderings. Indeed, this strategy finds its experimental realization in recent work done on atomic BEC systems in optical lattices [2], and our approach provides a unique theoretical guidance to achieve that goal.

There are some open issues. One regards the application of the HMFT approach to study fermionic problems, for example, a Hamiltonian like Eq. (1) but where the operators $\bar{b}^{\dagger}_{\mathbf{i}\sigma}$ for different modes on a lattice $\mathbf{i}$ anticommute. The method could certainly be used, however, fermions do introduce a non-local gauge potential [10] leading to an effective dynamical frustration which is difficult to handle in a controlled manner. Another issue concerns the application of the HMFT method when longer-range interactions are involved. There is already evidence from work on the J_1-J_2 $SU(2)$ Heisenberg model [19] that our HMFT will work in those cases as far as homogeneous phases are concerned. Actually, the SW MF theory introduced by Arovas and Auerbach [20] in the fundamental representation is a particular case of our general HMFT. Finally, our methodology exhausts all broken symmetry instances but it is still quite possible to have purely topological quantum orders and their corresponding phase transitions which cannot be described by broken symmetries and associated OPs [21] and, thus, are not included in our framework.

Summarizing, we developed a theoretical framework and a calculational scheme to study coexistence and competition of thermodynamic phases in strongly correlated matter. In our method (given a Hamiltonian modeling the physical system) the order parameters are not guessed but rigorously determined from group theoretical considerations as symmetry generators of a hierarchical language. In this way, the Hamiltonian operator (which does not necessarily have the full symmetry of the hierarchical group) is expressed in terms of symmetry generators. Then, in a non-phenomenological approach dubbed *hierarchical mean-field theory*, we approximated the dynamics (and thermodynamics) treating all possible local order parameters on an equal footing, i.e., without preferred symmetry axis. One could say that this procedure follows the guiding principles of *maximum symmetry* and *minimum information*. This allowed us to obtain in a simple manner the phase diagram of a model problem exhibiting coexistence and competition between antiferromagnetism and superfluidity. Combined with an analysis of the fluctuations (to analyze the stability of the mean-field) one now has a simple machinery to design phase diagrams.

This work was sponsored by the US DOE. We thank J.E. Gubernatis for a careful reading of the manuscript.

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[11] We know that for $J = V = 2t = \frac{2\bar{\mu}}{\mathbf{z}} < 0$, we can find the exact ground and lowest energy states of H [5].

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[14] If we are looking for solutions which could break the lattice translational symmetry (inhomogeneous), we should consider OPs $\mathcal{S}^{\mu\nu}(\mathbf{k}) = \frac{1}{N_s}\sum_{\mathbf{j}} e^{i\mathbf{k}\cdot\mathbf{r}_{\mathbf{j}}}\,\mathcal{S}^{\mu\nu}(\mathbf{j})$, for all possible $\mathbf{k}$-vectors. In general, non-local OPs can be derived (starting from our local classification) similarly to what has been done in the literature. For instance, for a quantum $S = 1$ system there are two possible local OPs: the local magnetization and the local spin-nematic parameters. Therefore, in the case of a quantum spin ($S = 1$) glass there will be two types of glasses: One where the local magnetization is frozen at each site and another where the local spin-nematic order is frozen at each site. In other words, there will be two Edwards-Anderson OPs.

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[16] Find a matrix T^{-1} such that $(T^{\dagger})^{-1}\tilde{H}T^{-1} = \mathrm{diagonal}$, and T satisfies the para-unitarity condition $T^{\dagger}\mathbb{I}T = T\mathbb{I}T^{\dagger} = \mathbb{I}$ with $\mathbb{I} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}$. The necessary and sufficient condition for an Hermitian matrix to be para-unitarily diagonalized (with all diagonal elements positive) is that the matrix be positive definite.

[17] In this case the eigenvalues can be determined in closed analytic form ($p = 0, 1, 2$)

$$
\begin{aligned}
(\omega_{p\mathbf{k}})^2 &= \epsilon + \sqrt{(\epsilon - (\lambda-\mu)^2)^2 + 16\mu\Lambda_B^2\frac{\lambda-\mu}{3}}\,\cos\left(\frac{\phi + 2p\pi}{3}\right) \\
\epsilon &= \frac{2}{3}(\lambda^2 + \frac{1}{2}(\lambda-\mu)^2 - 2\Lambda_B^2 - \Lambda_A^2) \\
\cos\phi &= \frac{2\eta(\eta^2 + \Lambda_B^2(\Lambda_A^2 - \Lambda_B^2 - \mu^2)) - 2\Lambda_B^2(\Lambda_A^2 + \mu^2)}{\sqrt{((\Lambda_B^2 - \eta)^2 - \frac{4}{3}\Lambda_B^2\mu(\mu-\lambda))^3}} \\
3\eta &= \Lambda_A^2 - \Lambda_B^2 + \mu(\mu - 2\lambda)
\end{aligned}
$$

[18] In two space dimensions, $\mathbf{z} = 4$, $\mathbf{k} = (k_x, k_y)$, $\mathbf{Q} = (\pi, \pi)$ and $\gamma_{\mathbf{k}} = (\cos k_x + \cos k_y)/2$.

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